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# Nonlinear Waves on a Rotating Viscous Fluid with a Cylindrical Free Surface*

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SUMMARY

An asymptotic theory is developed for the study of nonlinear wave motion of a rotating viscous fluid with a cylindrical free surface. The method used here is based upon a multiple-parameter singular perturbation scheme within the framework of long-wave approximation. Wave speed and a set of asymptotic evolution equations are derived, and a criterion for the instability of the wave motion is defined.

## 1. Introduction

In recent years there has been growing interest in the study of motions of a viscous fluid with free surface. Notably a large number of papers on a viscous fluid flow down an inclined plane and similar problems have appeared (Benjamin [1], Yih [2], Benney [3], Mei [4] and many others). In this paper we shall consider nonlinear wave motion of a viscous fluid with a cylindrical free surface. The fluid is assumed to be supported by a rotating circular cylinder and pulled toward the axis of the cylinder by a constant body force, and on the free surface a tangential stress is prescribed. Such a mathematical model is of geophysical significance, and may be relevant to long waves generated by wind, propagating along a latitude on the ocean or on a layer of the atmosphere, where the viscosity of the fluid plays an indispensable role. Although the fluid motion considered here is still assumed to be two-dimensional, our approach is different from those used previously. We shall refrain from formulating the problem in terms of a stream function, the existence of which, needless to say, relies upon the fluid motion being two-dimensional. Hence, the method developed in this paper can readily be generalized to three-dimensional problems (Shen and Shih [5]).

The basic idea involved in our approach is based upon one conceived by Shen [6] to study internal waves in a stratified inviscid fluid supported by a circular cylinder. In that case, periodic waves of cnoidal type were found to rotate about the cylinder at a constant angular speed. For the problem studied here, the same conjecture is made that nonlinear waves may also appear following a rotating cylindrical system at a constant angular speed. We formulate the problem in section 2, and the wave speed and evolution equations governing the wave motion are respectively determined in sections 3 and 4. It is noted that we essentially consider here a small but nonlinear perturbation of a given equilibrium state. However, since our asymptotic scheme is rather general in scope, an evolution equation governing large-amplitude waves will also be derived in section 5. This unified approach may clarify some of the seemingly paradoxical nature of the singular perturbation method dealing with free surface motions of a viscous fluid. In section 6, we shall derive a criterion for the instability of the wave motion, and a discussion of the results is also given.

## 2. Formulation of the Problem

We consider an incompressible, viscous fluid of uniform density $\rho$ and depth $h$, supported by a rigid circular cylinder of radius $a$ and pulled toward the axis of the cylinder by a constant body force $g$. The cylinder is assumed to rotate at an angular speed $\omega^*$, and the force acting

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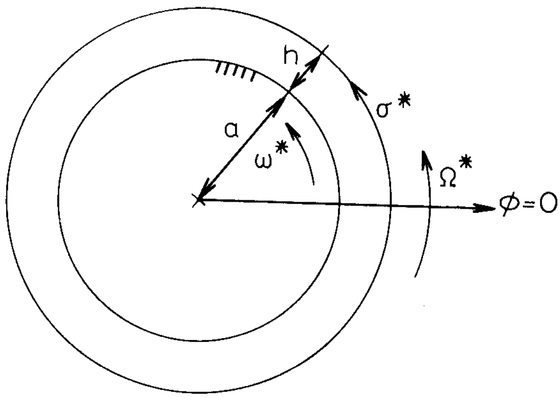


Figure 1. The rotating coordinate system.

on the free surface is represented by a tangential stress $\sigma^*$ perpendicular to the direction of the axis. It is assumed that a surface wave has been generated and we choose a cylindrical system $(r^*, \phi^*)$ rotating at a constant speed $\Omega^*$ to observe the wave motion (Fig. 1). In reference to this rotating coordinate system, the equations governing the fluid motion are (Landau and Lifschitz [7])

$$(r^* U^*)_{r^*} + V^*_\phi = 0\,, \tag{1}$$

$$\begin{aligned}\rho[U^*_{\tau^*} + U^* U^*_{r^*} + (r^*)^{-1} V^* U^*_\phi - (V^*)^2/r^* - 2\Omega^* V^* - (\Omega^*)^2 r^*] \\ = -p^*_{r^*} - \rho g + \mu[U^*_{r^*r^*} + U^*_{\phi\phi}(r^*)^{-2} + U^*_{r^*}/r^* - 2V^*_\phi (r^*)^{-2} - U^*(r^*)^{-2}]\,,\end{aligned} \tag{2}$$

$$\begin{aligned}\rho[V^*_{\tau^*} + U^* V^*_{r^*} + V^* V^*_\phi/r^* + U^* V^*/r^* + 2\Omega^* U^*] \\ = -p^*_\phi/r^* + \mu[V^*_{r^*r^*} + V^*_{\phi\phi}(r^*)^{-2} + V^*_{r^*}/r^* + 2U^*_\phi(r^*)^{-2} - V^*(r^*)^{-2}]\,,\end{aligned} \tag{3}$$

subject to the conditions

$$\zeta^*_{\tau^*} - U^* + V^* \zeta^*_\phi/r^* = 0\,, \tag{4}$$

$$p^* - 2\mu U^*_{r^*} + \mu(U^*_\phi/r^* + V^*_{r^*} - V^*/r^*)\zeta^*_\phi/r^* = \sigma^* \zeta^*_\phi/r^*\,, \tag{5}$$

$$\mu(U^*_\phi/r^* + V^*_{r^*} - V^*/r^*) + [p^* - 2\mu(V^*_\phi/r^* + U^*/r^*)]\zeta^*_\phi/r^* = \sigma^*\,, \tag{6}$$

at the free surface $r^* = a + h + \zeta^*(\phi, \tau^*)$, and

$$U^* = 0\,, \quad V^* = (\omega^* - \Omega^*)a\,, \tag{7}$$

at the surface of the cylinder $r^* = a$, where $(U^*, V^*)$ is the velocity vector, $\tau^*$ the time, $p^*$ the pressure, $\mu$ the constant viscosity coefficient, and for simplicity $\sigma^*$ is assumed to be constant and surface tension neglected.

To non-dimensionalize the equations, we define

$$\begin{aligned}
&r = r^*/a, && \tau = \tau^*(g/a)^{\frac{1}{2}}, && U = U^*(ga)^{-\frac{1}{2}}, && V = V^*(ga)^{-\frac{1}{2}},\\
&\Omega = \Omega^*(g/a)^{\frac{1}{2}}, && \zeta = \zeta^*/a, && \omega = \omega^*(g/a)^{\frac{1}{2}}, && p = p^*(\rho g a)^{-1},\\
&\sigma = \sigma^*\mu^{-1}(g/a)^{\frac{1}{2}}, && R = a\rho\mu^{-1}(ga)^{\frac{1}{2}}, && b = (a+h)/a.
\end{aligned}$$

For the purpose of constructing a stretching transformation, a class $F$ of continuous functions $f(\varepsilon)$ is introduced, which possess the following properties:
(1) $f(\varepsilon) > 0$ for $0 < \varepsilon \leqq \varepsilon_0$ and $\lim_{\varepsilon \to 0} f(\varepsilon) = 0$.
(2) For any $f, g \in F$, we say $f = g$ if $\lim_{\varepsilon \to 0} f/g = M > 0$ where $M$ may be set equal to $1, f > g$ if $\lim_{\varepsilon \to 0} f/g = \infty$, and $f < g$ if $\lim_{\varepsilon \to 0} f/g = 0$. Between $f$ and $g$ one of the relations defined should hold.

By means of the elements in $F$, we introduce the following stretching transformation

$$t=\beta\tau\,,\quad \theta=\alpha\phi\,,\quad u=\alpha^{-1}U\,,$$

where $\alpha$, $\beta\in F$. The stretching of $\tau$, $\phi$ and $u$ reflects the following considerations. We wish to study long-wave motions at large time, and the magnitude of the radial velocity component is assumed to be small. However, we use the same parameter $\alpha$ to stretch both $u$ and $\phi$ so that the equation of continuity is invariant under the stretching transformation. It is also noted that the parameters $R$, $\omega$, $\sigma$, $b$ and $\Omega$ are all assumed to be independent of $\varepsilon$. We write $v$, $\eta$ respectively for $V$ and $\zeta$. In terms of the new variables, (1) to (7) become

$$(ru)_r+v_\theta=0\,, \tag{8}$$

$$\begin{aligned}&\beta\alpha u_t+\alpha^2uu_p+\alpha^2r^{-1}vu_\theta-v^2r^{-1}-2\Omega v-\Omega^2r\\&=-p_r-1+R^{-1}(\alpha u_{rr}+\alpha^3r^{-2}u_{\theta\theta}+\alpha u_rr^{-1}-2\alpha v_\theta r^{-2}-\alpha ur^{-2})\,,\end{aligned} \tag{9}$$

$$\begin{aligned}&\beta v_t+\alpha uv_r+\alpha vv_\theta r^{-1}+\alpha uvr^{-1}+2\alpha\Omega u\\&=-\alpha p_\theta r^{-1}+R^{-1}(v_{rr}+\alpha^2v_{\theta\theta}r^{-2}+v_rr^{-1}+2\alpha^2u_\theta r^{-2}-vr^{-2})\,,\end{aligned} \tag{10}$$

$$\left.\begin{aligned}&\beta\eta_t+\alpha(-u+v\eta_\theta r^{-1})=0 && (11)\\&Rp-2\alpha u_r+(\alpha^2r^{-1}u_\theta+v_r-vr^{-1}-\sigma)\alpha\eta_\theta r^{-1}=0 && (12)\\&\alpha^2u_\theta r^{-1}+v_r-vr^{-1}-\sigma+[Rp-2\alpha r^{-1}(v_\theta+u)]\alpha\eta_\theta r^{-1}=0 && (13)\end{aligned}\right\}\quad \text{at } r=r+\eta\,,$$

$$u=0\,,\quad v=\omega-\Omega \qquad \text{at } r=1\,. \tag{14}$$

Assume that $u$, $v$, $p$ and $\eta$ possess asymptotic expansions

$$\left.\begin{aligned}u&=u_0+\delta_1u_1+\delta_2u_2+\ldots+\delta_nu_n\,,\\v&=v_0+\delta_1v_1+\delta_2v_2+\ldots+\delta_nv_n\,,\\p&=p_0+\delta_1p_1+\delta_2p_2+\ldots+\delta_np_n\,,\\\eta&=\eta_0+\delta_1\eta_1+\delta_2\eta_2+\ldots+\delta_n\eta_n\,,\end{aligned}\right\} \tag{15}$$

where $\delta_j\in F$, $\delta_{j+1}<\delta_j$. Substitution of (15) in (8) to (14) will yield a sequence of equations and boundary conditions, and different asymptotic theories are obtained depending upon $\alpha>\beta$ or $\alpha=\beta$. A detailed explanation of these cases and others may be found in Shen [8]. In what follows, we shall be mainly concerned with the more difficult case $\alpha>\beta$, that is, the time scale is much larger than the length scale in the direction of wave propagation. The case $\alpha=\beta$ corresponds to large-amplitude wave motions and will be discussed in section 5.

## 3. Asymptotic Motion of Small-Amplitude Waves—Zeroth and First Approximations

We assume that $\alpha>\beta$, $u_0=\eta_0=0$ and $v_0$, $p_0$ are functions of $r$. From (8) to (14), $v_0$, $p_0$ satisfy

$$-v_0^2r^{-1}-2\Omega v-\Omega^2r=-p_{0r}-1\,, \tag{16}$$

$$v_{0rr}+v_{0r}r^{-1}-v_0r^{-2}=0\,, \tag{17}$$

$$\left.\begin{aligned}&p_0=0 && (18)\\&v_0r-v_0b^{-1}-\sigma=0 && (19)\end{aligned}\right\}\quad \text{at } r=b\,,$$

$$v_0=\omega-\Omega \qquad \text{at } r=1\,. \tag{20}$$

It is readily found from (16) to (20) that

$$v_0=(\omega-\Omega)r+\sigma b^2(r-r^{-1})/2\,, \tag{21}$$

$$p_{0r} = r^{-1}[\omega r + \sigma b^2(r - r^{-1})/2]^2 - 1 . \tag{22}$$

From (18), it follows that

$$p_0 = (b-r) - (b^2 - r^2)[(\omega + \sigma b^2/2)^2 + \sigma^2 b^2/(4r^2)]/2 + \sigma b^2(\omega + \sigma b^2/2) \ln (b/r) .$$

The equations for the first approximation are

$$(ru_1)_r + v_{1\theta} = 0 , \tag{23}$$

$$2v_0 v_1 r^{-1} + 2\Omega v_1 = p_{1r} , \tag{24}$$

$$v_{1rr} + v_{1r} r^{-1} - v_1 r^{-2} = 0 , \tag{25}$$

$$\left.\begin{aligned} &-u_1 + b^{-1} v_0 \eta_{1\theta} = 0 \qquad (26)\\ &p_1 + p_{0r}\eta_1 = 0 \qquad (27)\\ &(v_{0rr} - v_{0r} b^{-1} + v_0 b^{-2})\eta_1 + v_{1r} - v_1 b^{-1} = 0 \qquad (28) \end{aligned}\right\} \quad \text{at } r = b ,$$

$$u_1 = 0 , \quad v_1 = 0 \qquad \text{at } r = 1 . \tag{29}$$

The general solution of (25) is

$$v_1 = f_1(\theta, t) r + g_1(\theta, t) r^{-1} . \tag{30}$$

From (21) and (28) to (30), we have

$$g_1 = -\sigma b \eta_1 , \quad f_1 = \sigma b \eta_1 , \quad v_1 = \sigma b \eta_1 (r - r^{-1}) . \tag{31}$$

We integrate (23) with respect to $r$ from $r=1$ to $r=b$, make use of (26), (29) and (31), and obtain

$$\eta_{1\theta}\left[(v_0)_{r=b} + \sigma b \int_1^b (r - r^{-1}) dr\right] = 0 .$$

Assume that $\eta_{1\theta} \neq 0$, and we have, by (21),

$$\Omega = \omega + \sigma(b^2 - \ln b - 1) . \tag{32}$$

From (23), (24), (27) and (29), it is obtained that

$$u_1 = -\sigma b \eta_{1\theta} r^{-1}[(r^2 - 1)/2 - \ln r] , \tag{33}$$

$$p_1 = 2\sigma b \eta_1 \left[\left(\omega + \frac{\sigma b^2}{2}\right)(r^2 - b^2)/2 - (\omega + \sigma b^2) \ln (r/b) - \sigma b^2 (r^{-2} - b^{-2})/4\right] - P_{0r}(b)\eta_1 . \tag{34}$$

## 4. Asymptotic Motion of Small-Amplitude Waves—Second Approximation

To derive asymptotic equations governing the evolution of the free surface motion, we should proceed to the equations for the second approximation. Here the orderings among the small parameters need be taken into account. First consider the case $\alpha = \delta_1$, $\delta_2 = \beta = \delta_1^2$, which leads to an equation similar to Burgers', and we have, from (8) to (14), the equations for the second approximation

$$(ru_2)_r + v_{2\phi} = 0 , \tag{34}$$

$$-(v_1^2 r^{-1} + 2v_0 r^{-1} + 2\Omega v_2) = -p_{2p} + R^{-1}(u_{1rr} + u_{1r} r^{-1} - 2v_{1\theta} r^{-1} - u_1 r^{-2}) , \tag{35}$$

$$u_1 v_{0r} + v_0 v_{1\theta} r^{-1} + u_1 v_0 r^{-1} + 2\Omega u_1 = -p_{1\theta} r^{-1} + R^{-1}(v_{2rr} + r^{-1} v_{2r} - v_2 r^{-2}) , \tag{36}$$

$$\left.\begin{aligned}&\eta_{1t}-u_{1r}\eta_1+\eta_{1\theta}b^{-1}(v_1+v_{0r}\eta_1-b^{-1}\eta_1 v_0)+v_0\eta_{2\theta}b^{-1}=u_2 &(37)\\ &R(p_2+p_{1r}+p_{0r}\eta_2+p_{0rr}\eta_1^2/2)-2u_{1r}=0 &(38)\\ &v_{2r}-b^{-1}v_2+\eta_2(v_{0rr}-v_{0r}b^{-1}+v_0 b^{-2})+\eta_1(v_{1rr}-v_{1r}b^{-1}+v_1 b^{-2})\\ &-\eta_1^2(b^{-3}v_0-b^{-2}v_{0r}+b^{-1}v_{0rr}/2-v_{0rrr}/2)=0 &(39)\end{aligned}\right\}\text{ at } r=b\,,$$

$$u_2=v_2=0 \qquad \text{at } r=1\,. \tag{40}$$

By (21), (30), (33) and (34), we may write (36) as

$$v_{2rr}+r^{-1}v_{2r}-v_2 r^{-2}=Rv_1(r)\eta_{1\theta} \tag{41}$$

where

$$\begin{aligned}&v_1(r)=\sigma^2 A(r)+\sigma\omega B(r)-\omega^2 br^{-1}+r^{-1}\,,\\ &A(r)=br^{-1}\left[(-b^2+\ln b+1)r^2-b^2\ln r-b^2r^{-2}-\frac{3b^4}{4}+b^2(3+2\ln b)-\ln b-\tfrac{3}{4}\right],\\ &B(r)=br^{-1}(r^2-2b^2+2\ln b+2)\,.\end{aligned} \tag{42}$$

The general solution of (41) may be expressed as

$$v_2=f_2(\theta,t)r+g_2(\theta,t)r^{-1}+I(r)\eta_{1\theta}\,, \tag{43}$$

where

$$I(r)=(R/2)\int_1^r (r-r^{-1}\xi^2)v_1(\xi)d\xi\,. \tag{44}$$

By (39), (40) and (43), we obtain

$$f_2+g_2=0\,,$$

$$f_2=[bI(b)-b^2I'(b)]\eta_{1\theta}/2+\sigma b\eta_2-\sigma\eta_1^2/2\,,$$

where $I'(r)=dI(r)/dr$, and it follows from (43) that

$$v_2=[(r-r^{-1})(bI(b)-b^2I'(b))/2+I(r)]\eta_{1\theta}+(r-r^{-1})(\sigma b\eta_2-\sigma\eta_1^2/2)\,. \tag{45}$$

We now integrate (34) with respect to $r$ from $r=1$ to $r=b$, make use of (37), (40) and (45), and obtain, at $r=b$,

$$\begin{aligned}bu_2&=b\eta_{1t}-bu_{1r}\eta_1+\eta_{1\theta}(v_1+v_{0r}\eta_1-b^{-1}\eta_1 v_0)+v_0\eta_{2\theta}\\ &=-\int_1^b\{[(r-r^{-1})(bI(b)-b^2I'(b))/2+I(r)]\eta_{1\theta\theta}+(r-r^{-1})(\sigma b\eta_{2\theta}-\sigma\eta_1\eta_{1\theta})\}\,dr\,,\end{aligned}$$

that is,

$$b\eta_{1t}+\sigma(b^2+2\ln b)\eta_1\eta_{1\theta}+(R/32)(A_1\omega^2+B_1\sigma\omega+C_1\sigma^2+D_1)\eta_{1\theta\theta}=0\,, \tag{46}$$

by (21), (31), (33), (42) and (44), where

$$\begin{aligned}A_1&=4b^5-16b^3\ln b-4b\,,\\ B_1&=6b^7-b^5(36\ln b+5)+b^3[32(\ln b)^2+32\ln b-10]+b(8\ln b+9)\,,\\ C_1&=5b^9-b^7(22\ln b+19)+b^5[24(\ln b)^2+75\ln b+1]\\ &\quad-b^3[24(\ln b)^2+6\ln b-17]-b(3\ln b+2)\,,\\ D_1&=-4b^4+16b^2\ln b+4\,,\end{aligned}$$

and we note that the coefficient of $\eta_{2\theta}$ vanishes because of (32).

Two more cases of interest should also be considered; the details of derivation of the evolution equations are the same as before and will be omitted.

(1) $\beta=\delta_2,\ \delta_2>\alpha^2,\ \alpha=\delta_1$
This case yields one-dimensional diffusion equation

$$b\eta_{1t}+(R/32)(A_1\omega^2+B_1\sigma\omega+C_1\sigma^2+D_1)\eta_{1\theta\theta}=0\,. \tag{47}$$

(2) $\beta=\alpha\delta_1,\ \delta_2=\delta_1^2,\ \delta_1>\alpha$
The second order derivative in (46) disappears, and we have

$$b\eta_{1t}+\sigma(b^2+2\ln b)\eta_1\eta_{1\theta}=0\,. \tag{48}$$

## 5. Asymptotic Motion of Large-Amplitude Waves—Zeroth Approximation

For large-amplitude waves, we let $\alpha=\beta$ and assume that $u_0$, $v_0$, $p_0$ and $\eta_0$ are all time-dependent. In this case, we need not use a rotating frame and may set $\Omega=0$. As $\varepsilon\to 0$, we have, from (8) to (15), the equations for the zeroth approximation

$$(ru_0)_r+v_{0\theta}=0\,, \tag{49}$$

$$-v_0^2r^{-1}=-p_{0r}-1\,, \tag{50}$$

$$v_{0rr}+v_{0r}r^{-1}-v_0r^{-2}=0\,, \tag{51}$$

$$\left.\begin{aligned}&\eta_{0t}-u_0+v_0\eta_{0\theta}(b+\eta_0)^{-1}=0 &(52)\\ &p_0=0 &(53)\\ &v_0r-v_0(b+\eta_0)^{-1}=\sigma &(54)\end{aligned}\right\}\quad \text{at } r=b+\eta_0\,,$$

$$u=0\,,\qquad v_0=\omega \qquad\qquad \text{at } r=1\,. \tag{55}$$

The general solution of (51) is

$$v_0=f_0(\theta,t)r+g_0(\theta,t)r^{-1} \tag{56}$$

By (54) and (55), we have

$$f_0+g_0=\omega\,,\qquad -2g_0(b+\eta_0)^{-1}=\sigma\,,$$

and it follows from (56) that

$$v_0=[\omega+\sigma(b+\eta_0)^2/2]r-\sigma(b+\eta_0)^2/(2r)\,. \tag{57}$$

We integrate (49) with respect to $r$ from $r=1$ to $r=b+\eta_0=\xi$, make use of (52), (55) and (57), and obtain, at $r=\xi$,

$$u_0\xi=\xi\eta_{0t}+v_0\eta_{0\theta}=-\int_1^\xi v_{0\theta}dr\,,$$

that is,

$$\xi_t+\xi_\theta(\omega+\sigma\xi^2-\sigma\ln\xi-\sigma)=0\,. \tag{58}$$

## 6. Discussion

So far we have achieved a unified approach to the derivation of various asymptotic equations, each of which, under appropriate conditions, describes certain stage of the evolution of a surface wave on the rotating fluid. The results may be summarized as follows. For $\alpha>\beta$, the asymptotic equations for small-amplitude waves are obtained, and the angular speed of a wave is near the value of the angular speed $\Omega$ given by (32). The solution methods of (46) to (48) are well-known (Hopf [9], Cole [10]), and a detailed study of periodic solutions of (46) may be found in Rosenblatt [11]. The case $\alpha=\beta$ leads to an asymptotic equation for large-amplitude waves. It generally does not possess a continuous periodic solution for all time, hence indicates the breaking of a surface wave. A discussion of such equation may be found in Courant and Hilbert [12]. Indeed, equation (48) also exhibits the same property.

Besides wave breaking, a viscous flow may not always remain laminar. Can the asymptotic theory yield a criterion for the instability of the wave motion? This question is answered by making the following observation. It is known that the initial value problems for (46) and (47) are not well-posed if the coefficient $(R/32)\,Q(\omega, \sigma, b)=(R/32)(A_1\omega^2+B_1\sigma\omega+C_1\sigma^2+D_1)$ of $\eta_{1\theta\theta}$ is of the same sign as that of $\eta_{1t}$, that is, $Q(\omega, \sigma, b)>0$, and this fact may be used to define a criterion for the instability of the wave motion. We say that the wave motion is asymptotically unstable if $Q(\omega, \sigma, b)\geqq 0$ and asymptotically stable if $Q(\omega, \sigma, b)<0$. For a fixed $b$, $Q(\omega, \sigma, b)=0$ defines a conic section in the $\omega$, $\sigma$-plane, and specifies the region of asymptotic stability of the wave motion. To illustrate this result, it is best to consider a special case. Assume $b=3$, and the condition $Q(\omega, \sigma, 3)=0$ yields

$$3\omega^2+16.10\omega\sigma+188.76\,\sigma^2=1\,,$$

that is,

$$(\omega'/0.55)^2+(\sigma'/0.073)^2=1\,,$$

where $(\omega', \sigma')$ are points in reference to a coordinate system obtained by rotating the $\omega$, $\sigma$-system by an angle of $-2.49°$. The wave motion is asymptotically stable for $(\omega, \sigma)$ in the interior of an ellipse, and asymptotically unstable otherwise. A plot of the ellipse is given in Fig. 2.

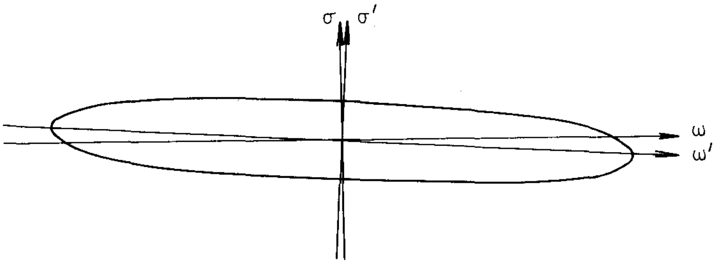


Figure 2. Region of asymptotic stability for $b=3$.

Finally we consider a simple example to see how a discontinuous periodic disturbance prescribed initially may be smoothed out by (46) with $Q(\omega, \sigma, b)<0$. Let $f(\theta, t)=(\sigma/b)$-$(b^2+2\ln b)\eta_1$. (46) in terms of $f$ becomes

$$f_t+ff_\theta=\nu f_{\theta\theta}\,, \tag{59}$$

where

$$\nu=-(R/32b)\,Q(\omega, \sigma, b)>0\,.$$

Assume that $f(\theta, 0)=4\nu\theta$, $-\pi<\theta<\pi$, with period $2\pi$ and $\int_{-\pi}^{\pi} f(\theta, 0)\,d\theta=0$, and we are looking for a continuous periodic solution of (59) satisfying the initial condition. Following [9], [10], we let

$$\varphi(\theta, t)=\exp\int_0^\theta(-1/2\nu)\,u(\phi, t)\,d\phi\,,$$

and have the following problem posed for $\varphi(\theta, t)$:

$$\varphi_t=\nu\varphi_{\theta\theta}\ \text{for}\ -\infty<\theta<\infty,\quad t>0\,,$$
$$\varphi(\theta, t)=\varphi(\theta+2\pi, t)\ \text{for}\ -\infty<\theta<\infty,\ t>0\,,$$
$$\varphi(\theta, 0)=\exp(-\theta^2)\ \text{for}\ -\pi<\theta<\pi\ \text{with period}\ 2\pi.$$

It is found that

$$\varphi = a_0/2 + \sum_{n=1}^{\infty} a_n \exp(-n^2 \nu t) \cos n\theta ,$$

where

$$a_n = (2/\pi) \int_0^{\pi} \exp(-\theta^2) \cos n\theta \, d\theta ,$$

and the solution of $f$ is given by

$$f = -2\nu\varphi_\theta/\varphi$$

$$= 2\nu \left[ \sum_{n=1}^{\infty} n a_n \exp(-n^2 \nu t) \sin n\theta \right] \left[ a_0/2 + \sum_{n=1}^{\infty} a_n \exp(-n^2 \nu t) \cos n\theta \right]^{-1}$$

It is easily shown that

$$\lim_{t\to 0} f(\theta, t) = 4\nu\theta \text{ pointwise for } -\pi < \theta < \pi \text{ with period } 2\pi.$$

For large $t$,

$$f \sim 4\nu a_0^{-1} a_1 \exp(-\nu t) \sin\theta ,$$

and the initial saw-tooth wave has been smoothed out to become a sinusoidal wave.

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